

This article just serves to record a proof I came up with of the classic (and very significant, see <https://arxiv.org/abs/2309.02070>) fact in the title. Whenever a result is cited (written in bold), it is from [Farley's article](#).

Setup: Let X be a locally finite-dimensional CAT(0) cube complex, and let $X^{(1)}$ be its 1-skeleton. We denote vertices by $u, v, x, y, z, \dots$, edges by $e, \dots$, cubes by $C, \dots$, and hyperplanes by $H, \dots$, and half-spaces by $\alpha, \dots$.

Lemma 0. Every interval

$$[x, y] := \{v : v \text{ lies on a geodesic from } x \text{ to } y\}$$

is convex.

Proof. By **Proposition 5.5** $v \in [x, y]$ iff v lies in every half-space $\alpha \supset \{x, y\}$, so $[x, y]$ is intersection of half-spaces. Half-spaces are convex by **Proposition 5.5**.

Lemma 1. If x, y are adjacent, then $d(x, z) = d(y, z) \pm 1$.

Proof. Immediately from the fact that $d(x, z)$ is the same as number of half-spaces containing x but not z .

Lemma 2. For every non-empty convex $A \subset X^{(1)}$ and vertex z , there is a unique $\text{proj}_A(z) \in A$ that minimizes the distance to z .

Proof. When $z \in A$ this is trivial. Otherwise if there are $x \neq y$ in A s.t. $d(x, z) = d(y, z) = l$, we claim by inducting on $d(x, y)$ that there is $v \in [x, y]$ s.t. $d(v, z) < l$:

When $d(x, y) = 1$ the assumption is impossible by Lemma 1. So base case is $d(x, y) = 2$. Let (x, u, y) be a geodesic. If $d(u, z) < l$ we are done; if not, let H_x (H_y resp.) be the hyperplane crossing (x, u) ((y, u) resp.). Then x, u, y, z lie in the four quadrants formed by H_x, H_y , as in the assumption of **Lemma 5.4**. Hence $H_x \cap H_y \neq \emptyset$. As (x, u, y) is a geodesic, $H_x \neq H_y$, so we can apply **Proposition 5.7** to obtain a square C spanned by x, u, y , and let v be the remaining vertex. As v lies in the same quadrant as z , we have $d(v, z) = l - 1 < l$.

For general $d(x, y) > 2$ we pick a geodesic $(x, x', \dots, y', y)$. If $d(x', z)$ or $d(y', z)$ is $l - 1$ then we are done. Otherwise $d(x', z) = d(y', z) = l + 1$, so by inductive hypothesis there is $v' \in [x', y']$ s.t. $d(v', z) \leq l$. If $d(v', z) < l$ we are done, so assume $d(v', z) = l$, and apply inductive hypothesis to the pair x, v' .

Hence the vertex in A minimizing distance to z must be unique.

Theorem 3. For every non-empty convex $A \subset X^{(1)}$ we have

$$d(z, x) = d(z, \text{proj}_A(z)) + d(\text{proj}_A(z), x).$$

Proof. When $z \in A$ this is trivial. Thus we assume $z \notin A$.

We claim by inducting on $d(x, v)$ that if $v = \text{proj}_{[x, v]}(z) \neq z$, then

$$d(z, x) = d(z, v) + d(v, x).$$

Base case that $d(x, v) = 0$ is trivial. For general $d(x, v) > 1$, pick a geodesic $(v, u, \dots, x)$, then

$$d(u, x) = d(v, x) - 1. \quad (1)$$

By Lemma 2 we have $d(z, u) \geq d(z, v) + 1$, while $d(z, u) \leq d(z, v) + d(v, u) = d(z, v) + 1$, so

$$d(z, u) = d(z, v) + 1. \quad (2)$$

Now by Lemma 2 again $u = \text{proj}_{[x, u]}(z)$, so by inductive hypothesis $d(z, x) = d(z, u) + d(u, x)$, so using equations (1) and (2) we get $d(z, x) = d(z, v) + d(v, x)$.

Back to the original statement. Let $v = \text{proj}_A(z)$; as $[x, v] \subset A$, we have $\text{proj}_{[x, v]}(z) = v$, so $d(z, x) = d(z, v) + d(v, x)$.

Theorem 4. For every x, y, z , there is a unique *median* $m \in [x, y] \cap [x, z] \cap [y, z]$, i.e. $X^{(1)}$ is a median graph.

Proof. A median is exactly an element $m \in [x, y]$ satisfying $d(m, x) + d(m, z) = d(x, z)$ and $d(m, y) + d(m, z) = d(y, z)$. We claim that a median m must be $\text{proj}_{[x, y]}(z)$.

By Theorem 3 we see that $\text{proj}_{[x, y]}(z)$ is indeed a median. Thus a median m satisfies

$$2d(m, z) = d(x, z) + d(y, z) - d(x, y) = 2d(\text{proj}_{[x, y]}(z), z).$$

The result then follows from Lemma 2.